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City Size Distribution as a Probability Process

Introduction

THE rapid growth of population has become a matter of general concern. Perhaps a more important aspect of this population growth is not just the change in size of the total number but an ever increasing unplanned concentration of population in special areas regarded as urban. The tempo and dimension of urbanization in recent years have never been equalled before in human history and this may continue for years to come. Over a fifteen or twenty year period, the demographic situtation of a country is affected to a greater degree by population redistribution than by its growth.

The existence of difference between population living in small villages and those living in big towns has been recognised for millenia. Size differences among urban centres are frequently as significant as the differences between rural and urban populations. There is no valid statistical evidence to show that a homogenization of urban life is taking place among different city size classes. It has been amply demonstrated that greater demographic differences exist between different size of cities than between rural and urban sectors each taken as a whole.

Urbanization is a process of concentration. It proceeds in both ways, by increasing the number and size of urban centres. A dichotomy of population into rural-urban is one among many measurements of population distribution and this alone may not provide all the details that are germane for an adequate understanding of an urban system. As a rule proportion of urban population is used in most comparative analysis to represent the extent of urbanization in a country. The classification of towns and cities into multiple size categories can be regarded as an extension of dichotomous classification and is regarded as

City Size Distribution (CSD) in demographic analysis. As demographic transition proceeds, changes in size and in the number of points of concentration take place. Size and its associated characteristics of population are as important between smaller and larger urban areas as between rural and urban areas. Statistics on the number of people in each size category then tell a great deal more about the settlement pattern in a country or region than data on the total urban population.

Richardson (1973) in a review of the theory of the city size distribution observed that it has not unfortunately been seriously analysed from the demographic point of view and remarked that urban analysis needs a demographic bias. He observed that the theoretical challenges of the problem remain immense and said although Markov Chain models have been suggested as providing useful approach to city size distribution, their potential has not been fully developed. He states the following as probable areas for further investigation :

1. Markovian representation of city size distribution
2. How the city size distribution varies among rapidly growing populations versus slowly growing populations
3. How variation in city size distribution can be explained at different levels of urbanization
4. Whether city size distribution can be thought of as a function of age distribution of cities

An attempt is made here to use the theory of Markov Chain as an analytical tool in the study of city size distribution. The details of Markov theory as applicable to a closed model are tested with 'Data Analysis' on urbanization in Tamil Nadu.

City Size Distribution as Markov Model

Let us consider an urban population whose cities are divided into k size classes. Let Ci (t) denote the number of cities in size class j at time t ($t = 0, 1, 2, \ldots$). The initial size class C_j (0), $(j = 1, 2, \ldots, k)$ are assumed to be given and we know that

$$C(t) = \sum_{j=1}^{k} C_j(t). \tag{1}$$

For $t > 0$, the size classes are random variables and we are mainly concerned with their expectations.

The conditional probability that a city moves to class; at time /, given that it was in class i at time t - 1, is regarded as the transition probability and is denoted by p_{ij}. The transition matrix for a urban process having j states ($j = 1, 2, \ldots, k$) with transition probabilities p_{ij} ($i, j = 1, 2, \ldots k$) can conveniently be represented in a matrix :

$$P = \begin{bmatrix} p_{11} \cdots p_{1j} \cdots p_{1k} \\ \cdot \; \cdots \; \cdots \; \cdot \\ p_{i1} \cdots p_{ij} \cdots p_{ik} \\ \cdot \; \cdots \; \cdots \; \cdot \\ p_{k1} \cdots p_{kj} \cdots p_{kk} \end{bmatrix} \tag{2}$$

If C_{ij} denote the number of cities moving from class i to j from time t to time $(t+1)$, then the transition probabilities are estimated from :

$$p_{ij} = C_{ij} / \sum_{j=1}^{k} C_{ij} \tag{3}$$

subject of course to the restrictions that $p_{ij} \geqslant 0$, for all i and j and $\sum_{j} p_{ij} = 1$ which are satisfied since $C_{ij} \geqslant 0$. The estimates of p_{ij}'s can be described as follows : Let C_{ij} for a given t be entered in a two-way table. The (i, j-th entry in the table divided by the sum of the entries of the ith row gives the estimates of p_{ij}. If the transition probability p_{ij} depends on the class i and j and not on time, then the conditional probability is called homogeneous. In addition to the time homogeneity, as an additional restriction to the model, we are not concerned about the entries and exits to the system and try to concentrate on a system which is closed for entries and exits. Hence as a first step time homogeneous discrete closed finite Markov Chain has been detailed to describe the city size distribution. Given the initial city size distribution $C_j(0), j = 1, 2, \ldots k$ and the transition matrix P, we obtain the expected size classes at successive points in time t ($t = 1, 2, \ldots, T$).

$$C_j(t+1) = \sum_{i=1}^{k} C_i(t)\, p_{ij} \tag{4}$$

or in matrix notation

$$\mathbf{C}(t+1) = \mathbf{C}(t)\, P \tag{5}$$

where $\mathbf{C}(t) = (C_1(t), C_2(t), \ldots, C_k(t))$ represents distribution of cities in k size classes at time t. Repeated application yields :

$$\mathbf{C}(t) = \mathbf{C}(0)\, P^t. \tag{6}$$

It is thus clear that a random process that results in a sequence of states that are Markovian with stationary transition probabilities is completely specified, when we know the initial probability vector and the transition probability matrix. It is well known that for any matrix of kind P, the ultimate distribution of $C(t)$ tends in time to a limiting distribution C, which is independent of the initial probability distribution C(0). The limiting distribution is called the equilibrium distribution, which satisfies the relationship :

$$\mathbf{C = CP.}$$

City Size Distribution as Absorbing Markov Chain

A preview of the Table 3 representing the transition probabilities for the data presented in this analysis shows that we are observing a special type of Markov chain wherein all the cities are likely to move to the largest class, F, however long the waiting time be. In cases where a Markov chain stays among transient states with positive probability, there is no long run distribution. In these cases the chain 'drifts to infinity' with a positive probability. The long run distribution of these chains even if it exists need not necessarily be a probability distribution. After projecting the city size distribution for a reasonable length of time, we may study other properties of the system, such as :

1. Given the process started in a state i what is the expected number of transitions before absorption?
2. What is the variance of the number of transitions before absorption, given the process started in a transient state?

The size classes through which cities pass may be of three types and the values in the matrix P define these three types. If every state can move to any other state and the system can be left, the system is transient and the values of one or more of horizontal lines in the matrix will total less than one. If every state can enter into any other state but the system cannot be left, the system is ergodic. In this case the total value of the horizontal line is one. Thirdly if one or more of the states are ergodic, the system is said to be absorbing and in this case there will be one or more horizontal lines in which one element will be equal to unity, rest being zero. The transition matrices constructed for the entire period 1901 to 1981 fall into this category of an absorbing Markov chain with $p_{FF} = 1$. If a city moves indefinitely, the movement is said to be unrestricted. However we can consider the movement of cities to be restricted in some ways, usually by the presence of a maximum size. For example a city starting at $X_0 = B$ may be restricted in such a way that the movement ceases once it reaches the state F. It is quite consistent with the classification of cities (states). It could be possible for us to extend the CSD beyond $F = 100{,}000+$, say by taking $F = 1{,}00{,}000$ to $10{,}00{,}000$ and $G = 10{,}00{,}000$ to $1{,}00{,}00{,}000$ and $H = 10{,}00{,}00{,}000+$. But we know that a city can increase its size only with reference to the spatial, socio-economic and demographic conditions. At some level, it may cease to grow beyond a class size (Lever, 1973) and can conveniently be assumed to be trapped into a state from where the movement ceases. Depending on international, national or regional analysis, the upper class size into which the cities can be trapped (United Nations, 1969) can be fixed unquestionably. We have taken here the census classification as reasonable and once the city reaches a size 100,000 it remains there for ever. It is quite likely, as evidence available elsewhere (Fuguitt, 1965) that a city after reaching the highest class may move backward. In that case the largest class need not be absorbing. But as we observe from the data on Tamil Nadu, no city reached a size of 100,000 has ever moved backward. On the other hand, the initial class is reflecting. A reflecting barrier of a CSD is defined as a state (class size) which when entered in a given direction, say upwards, holds the city until a positive jump occurs and allows the city to move up and resume the movement. For the type of data at our disposal a Markov chain with absorbing state in the highest class is appropriate to study the properties of the city size distribution.

City Size Distribution with Nonstationary Transition Matrices

The transition matrices constructed for Tamil Nadu during 1901-81, not only has the feature $p_{FF} = 1$ but also depict a tendency of having greater upward movements as time passes. The probabilities in the diagonal and lower diagonal elements decrease and the probabilities on the upper diagonal elements increase on the time scale. In this section we would like to represent the effect of relaxing the 'time homogeneous' assumption we had made earlier. The time homogeneous assumption may be reasonable in some populations but it is of interest to see that a generalization is straight forward even under nonhomogeneous (in time) condition.

Let us suppose that we have a series of transition matrices P_t $(t = 1, 2, \ldots, T)$, then the equation can be written under nonstationary assumption as :

$$C(t) = C(0)\, P(t) \text{ where } P(t) = P_1\, P_2 \ldots\ldots P_t \qquad (8)$$

The two equations (6) and (8) and their generalizations are the basis for most prediction exercise. The transition matrix P or the series P_t $(t = 1, 2, \ldots T)$ are usually estimated from current or historical data.

Projection of Transition Matrices

It is well known that under the assumption of stationarity, the projection of

CSD beyond the final point of observation is possible. On the other hand, under nonstationary assumption such a projection is impossible. In practical situation it will be of great interest to obtain estimates of CSD beyond the observed period. Transition probabilities will be useful to project CSD beyond the ultimate point of observation as much under nonstationary assumption as in the case of stationary process.

It is seen that P_{ii}'s are large up to 1931 indicating a slow movement of cities across size classes. From 1941 to 1981, there is a reduction in p_{ii}'s and an increase in forward movement of cities resulting from rapid growth of population. To overcome this difficulty three transition matrices were constructed based on the average movements of (1) 1901-31, (2) 1941-71 and (3) 1901-71. These matrices are better suited to study a stationary process compared to the transition matrices constructed for individual years (Anderson and Goodman, 1957). Because of the growth of population, a time dependent nonstationary assumption may produce better results.

Time dependences envisaged by Prais (1955) and Matras (1967) are different in nature. Prais postulated that the discrepency between class structure for fathers and sons might be due to the changes in the definition of classes. Matras suggested that the transition probabilities relating to movements between $t-1$ and t might be made functions of class structure at time $t-1$. But in the present investigation, we have kept the class sizes (size classes A to F) intact through 1901-'81. Hence changes in the transition matrices are due to heterogeneity in city size distribution.

The estimates for a nonstationary process do produce the desired results but projections (beyond data availability) require further assumptions. Some considerations on the transition matrices used to project CSD under nonstationary assumption are detailed here. After closely following the cell probabilities, a co-ordinate-wise linear projection is made, using

$$p_{ij}(t) = a_{ij} + b_{ij}(t) \qquad i,j = 1, 2, \ldots, k \quad t = 1, 2, \ldots, T \tag{9}$$

row sums being normalised to unity. The linear projection tried though useful in the short term, may lead to negative and unserviceable probabilities creating limitations in the use of this model. Being a stochastic matrix, increase in one or more cell probabilities of a given matrix in a given row should be compensated for by a decrease in one or more cell probabilities. A close look at the transition matrices reveals :

1. All the transition matrices depict a general raadom walk with movements to the next neighbouring classes;
2. $p_{i,i+1}$ is always greater than $p_{i+1,i}$;

3- pi, i-1 is always less than $p_{it\ i+1}$;
4. the diagonal elements decrease in value indicating greater movements as time passes;
5. the upper diagonal elements which were absent or very small in the initial stages gain momentum and absorb the entire decrease in the diagonal elements;
6. the lower diagonals which were present in the early periods behave in a hapazard way but certainly indicate a decline in their value as time passes.

Based on these observations, the projections of transition matrices were done under growth curve assumption. The constraints on the probability that it will never be greater than one or less than zero led to the use of logistic and modified exponential curves to build future probabilities The choice either logistic or exponential was done with the help of the growth rates using the curve :

$$p_{ij}(t) = a_{ij} b_{ij}^{t}, \qquad i, j = 1, 2, \ldots, k \quad t = 1, 2, \ldots, T \tag{10}$$

fitted to the observed data available with us. Here $bij > 1$ indicates the increase and $bij < 1$ indicates the decrease. The modified exponential fit

$$p_{ij}(t) = A \exp(-\theta t), \qquad t = T + 1, T + 2, \ldots. \tag{11}$$

is done whenever $bij < 1$, so that the probabilities will never be negative. The logistic fit

$$p_{ij}(t) = 1/(1 + A \exp(-\theta t), \qquad t = T + 1, T + 2, \ldots \tag{12}$$

is done whenever $bij > 1$, so that the probabilities can never be greater than one. These estimates are arrayed, row sums being normalised to one to obtain the future transition probability matrix.

Data Analysis

The data for this analysis are taken from Census of India, (1971), Tamil Nadu and Census of India, (1981), Tamil Nadu, Provisional Population Total. We are concerned about the representation of city size distribution by Markov model, and the first step is restricted to a closed model. The number of urban places are so selected to fit into the system of closed model. The urban places are classified into six classes, *A :* 'below 5,000'., *B :* 5,000 to 10,000., C: 10,000 to 20,000., *D :* 20,000 to 50,000 , *E :* 50,000 to 100,000 and F : above 100,000', The places included in the analysis are so selected that a place identified as urban should continue to hold the status of urban throughout the period of observation from 1901-1981. There are 116 such places holding a continuous status of urban during 1901-1981. This kind of selection implicitly provides an opportunity to demonstrate the use of Markov model to study the changes in the internal structure of a 'cohort of cities'. The cohort constituting the number of places continue to hold the status of urban during the entire period.

Places belonging to 'less than 5000' in size, continuously holding the status of urban during 1901-1981 are five in number and they have been omitted from the analysis in the closed model. Inclusion of places with size 'less than 5000' satisfying the criterion of closed model poses certain difficulties in the construction of transition probabilities. The analysis concerning the remaining 111 places constituting the closed model are presented here.

First, it was attempted to check up the assumption of the stationarity or otherwise of the process. This has been carried out with chi-square test discussed in Anderson and Goodman (1957) and Bhat (1972). In the case of closed model relating to 111 cities, the chi-square test gave a value of 69.2 with 31 degrees of freedom, this being significant at 5% level. Since the deviations are not considered to be of any serious nature, estimates of CSD under the stationary and nonstationary assumptions are provided.

Table 1 gives the distribution of urban places by size with population from 1901 to 1981 in five classes *B* to *F*. The number of places remain constant at 111 as indicated above with no addition or deletion being possible in the closed model. It can be seen that over a period of eighty years the percentage of places with size 20,000 and above has increased from 24 in 1901 to 84 in 1981 while the corresponding population increased from 62% in 1901 to 98% in 1981. The percentage of places with size below 20,000 has decreased from 76 in 1901 to 16 in 1981 while the corresponding population had declined from 38 to 2 per cent. This indicates the movements of cities from one size class to another and the Markov model is a very useful tool to summarise the changing internal structure of the city size distribution.

Table 2 gives changes in CSD obtained by cross tabulating the urban areas by size at the beginning and at the end of each decade. This forms the basis for the construction of the matrices of transition probabilities.

Table 3 shows the movements of urban places between size classes in terms of transition probabilities (decade wise). Each element in *P* gives the probability of moving between size classes during a given decade, diagonal elements representing the chance of continuing in the same size class. Each row of the matrix gives the probability of being in different size classes at the end of the decade given that they were in a given size class at the beginning of the decade. For example in the transition matrix $P_{1931-41}$, the diagonal element *PEE* — 0.7778 informs that a city found in size class *E* in 1931 has a probability of 0.7778 to continue in the same size class in 1941. *PED* 0.1111 gives the probability that a city found in class *E* in 1931 has a probability of 0.1111 to move backward

TABLE 1-DISTRIBUTJON OF URBAN PLACES BY SIZE
TAMILNADU : 1901-1981

Size class	*No. of cities*	*Population*	*Size class*	*No. of cities*	*Population*
	1901			1941	
B	36 (32)	2,80,017(11)	*B*	20 (18)	1,70,866 (4)
C	49 (44)	6,91,153 (27)	*C*	45 (41)	6,51,465 (17)
D	17(16)	4,81,813(19)	D	31 (28)	9,07,953 (23)
E	6(5)	3,50,650(14)	E	10 (9)	6,39,637 (16)
F	3(3)	7,51,872(29)	F	5 (4)	15,15,213 (40)
	111	25,55,505		111	39,53,081
	1911			1951	
B	31 (28)	2,51,131 (9)	B	10 (9)	82,765 (2)
C	49 (44)	6,94,167(25)	C	38 (34)	5,45,196 (10)
D	22 (20)	6,41,167(23)	D	45 (41)	14,27,808 (26)
E	6(5)	3,54,747 (13)	E	11(10)	8,04,292 (15)
F	3(3)	8,13,262 (30)	F	7(6)	26,03,552 (47)
	111	27.54,803		111	54,63,715
	1921			1961	
B	31 (28)	2,47,870 (9)	B	5(5)	38,992 (1)
C	48 (43)	6,75,381 (24)	C	39(35)	5,61,285 (9)
D	20(18)	5,50,537 (19)	D	40 (36)	12,71,570 (19)
E	9(8)	5,08,557 (18)	*E*	18 (16)	11,92,569(18)
F	3(3)	8,37,866 (30)	F	9(8)	33,94,541 (53)
	11!	28,20,211		111	64,58,957
	1931			1971	
B	22(20)	1,77,456 (5)	B	2(2)	16,069 (0)
C	47 (42)	6,49,045 (19)	*C*	29 (26)	4,58,145 (5)
D	29 (26)	8,11,557(24)	D	42 (38)	14.06,457 (17)
E	9(8)	5,75,447(17)	E	22 (20)	13.62,935 (16)
F	4(4)	11,40,434 (35)	F	16 (14)	53,47,736 (62)
	111	33,53,939		111	85,91,343
				1981	
			B	3(3)	24,079 (0)
			C	15(13)	2,38,361 (2)
			D	45(41)	14,66,724(13)
			E	30 (27)	20,19,776 (18)
			F	18(16)	74,62,480 (67)
				111	1,12,11,420

Figures in brackets indicate percentage.

TABLE 2—CHANGES IN CITY SIZE DISTRIBUTION TAMILNADU : 1901-1981

		B	*C*	*D*	*E*	*F*	*Total*
	1	2	3	4	5	6	7
1901-11	B	28	8				36
	C	3	41	5			49
	D			16	1		17
	E			1	5		6
	F					3	3
	Total	31	49	22	6	3	111
1911-21	B	26	5				31
	C	4	42	3			49
	D	1	1	17	3		22
	E				6		6
	F					3	3
	Total	31	48	20	9	3	111
1921-31	B	20	11				31
	C	2	36	10			48
	D			18	2		20
	E			1	7	1	9
	F					3	3
	Total	22	47	29	9	4	111
1931-41	B	17	5				22
	C	3	39	5			47
	D		1	25	3		29
	E			1	7	1	9
	F					4	4
	Total	20	45	31	10	5	111

Table 2 *(contd. on page* 26)

Table 2 *(contd. from page* 25)

		1	2	3	4	5	6	7
1941-51		B	9	11				20
		C	1	27	17			45
		D			28	3		31
		E				8	2	10
		F					5	5
	Total		10	38	45	11	7	111
1951-61		B	4	6				10
		C	1	32	5			38
		D		1	35	9		45
		E				9	2	11
		F					7	7
	Total		5	39	40	18	9	111
1961-91		B	2	3				5
		C		26	13			39
		D			29	11		40
		E				11	7	18
		F					9	9
	Total		2	29	42	22	16	111
1971-81		B	2					2
		C	1	15	13			29
		D			32	10		42
		E				20	2	22
		F					16	16
	Total		3	15	45	30	18	111

TABLE 3-TRANSITION MATRICES: TAMILNADU. 1901.'81

3 A-TRANSITION MATRICES OF OBSERVED MOVEMENTS

		B	C	D	E	F
$P_{1901-11}$	B	0.7778	0.2222			
	C	0.0612	0.8367	0.1021		
	D			0.9412	0.0588	
	E			0.1667	0.8333	
	F					1.0000
$A_{1911-21}$	B	0.8387	0.1613			
	C	0.0816	0.8572	0.0612		
	D		0.0477	0.8095	0.1428	
	E				1.0000	
	F					1.0000
$P_{1921-31}$	B	0.6452	0.3548			
	C	0.0417	0.7500	0.2083		
	D			0.9000	0.1000	
	E			0.1111	0.7778	0.1111
	F					1.0000
$A_{1931-41}$	B	0.7727	0.2273			
	C	0.0638	0.8298	0.1064		
	D		0.0345	0.8621	0.1034	
	E			0.1111	0.7778	0.1111
	F					1.0000
$P_{1941-51}$	B	0.4500	0.5500			
	C	0.0222	0.6000	0.3778		
	D			0.9032	0.0968	
	E				0.8000	0.2000
	F					1.0000
$P_{1951-61}$	B	0.4000	0.6000			
	C	0.0063	0.8421	0.1316		
	D		0.0222	0.7778	0.2000	
	E				0.8182	0.1818
	F					1.0000
$P_{1961-71}$	B	0.4000	0.6000			
	C		0.6667	0.3333		
	D			0.7250	0.2750	
	E				0.6111	0.3889
	F					1.0000
$P_{1971-81}$	B	1.0000				
	C	0.0345	0.5172	0.4483		
	D			0.7619	0.2381	
	E				0.9091	0.0909
	F					1.0000

3 B-ESTIMATED TRANSITION MATRICES OF AVERAGE MOVEMENTS

		B	*C*	*D*	*E*	*F*
1901-31	B	0.7539	0.2461			
	C	0.0615	0.8146	0.1239		
	D		0.0159	0.8836	0.1005	
	E			0.0926	0.8704	0.0370
	F					1.0000
$P_{1941-71}$	B	0.4167	0.5833			
	C	0.0162	0.7029	0.2909		
	D		0.0074	0.8020	0.1906	
	E				0.7431	0.2569
	F					1.0000
$P_{1901-71}$	B	0.6121	0.3879			
	C	0.0424	0.7689	0.1887		
	D		0.0149	0.8445	0.1396	
	E			0.0556	0.8026	0.1418
	F					1.0000

3 C—ESTIMATED TRANSITION MATRICES : LINEAR PROJECTION

		B	*C*	*D*	*E*	*F*
$P_{1971-81}$	B	0.2969	0.7031			
	C	0.1938	0.5381	0.2681		
	D		0,2806	0.5395	0:1799	
	E			0.0274	0.1705	0.8021
	F					1.0000
$P_{1981-91}$	B	0.2181	0.7819			
	C	0.0235	0.5916	0.3849		
	D		0.1253	0.6326	0.2421	
	E			0.4842	0.1264	0.3894
	F					1.0000
$P_{1991-2001}$	B	0.1393	0.8607			
	C	0.0132	0.6278	0.3590		
	D		0.0932	0.6315	0.2753	
	E			0.5691	0.0620	0.3689
	F					1.0000

3 D—ESTIMATED TRANSITION MATRICES : GROWTH CURVES

		B	*C*	*D*	*E*	*F*
$P_{1971-81}$	B	0.3235	0.6765			
	C		0.6597	0.3403		
	D			0.7414	0.2753	
	E				0.5230	0.4770
	F					1.0000
$P_{1981-91}$	B	0.2757	0.7243			
	C		0.6168	0.3832		
	D			0.7035	0.2965	
	E				0.4776	0.5224
	F					1.0000
$P_{1991-2001}$	B	0.2364	0.7636			
	C		0.5757	0.4243		
	D			0.6648	0.3352	
	E				0.4403	0.5597
	F					1.0000

to class *D* in 1941 and PEF = 0.1111 gives the probability of moving forward from class E *in* 1931 to class *F in* 1941. Other elements of the matrix can be given similar interpretation. A careful observation of the matrices reveals that each matrix has *PFF* = 1, rest of the elements being zero in the last row. So the *F* class becomes the absorbing class of the CSD. It is seen from the observed transition matrices through 1901-1981 that it is certain that all the cities at one point of time will reach the size class *F,* and remain there for ever.

Comparison of these eight transition probability matrices shows that the probabilities of movements between size classes are not similar from decade to decade. The tendency to move froward to larger size classes is greater in the later three (1951-81) than during the earlier three (1901-31) decades. The most important feature of the transition matrices is that they depict growth rather than decline. This can be demonstrated more clearly through Markov model, by determining the consequences by treating each of these transition matrices as constant extending over a period of time.

The difficulties of treating each of the transition matrix as constant can be overcome by constructing transition matrices by grouping the time periods into homogeneous groups. Three such transition matrices were constructed based on the average movements between 1901-31, 1941-71 and 1901-71 and are presented in Table 3b. The following procedure has been adopted to project the transtion matrices beyond 1971 closely following the methods described. In the case of a closed model, the nature and behaviour of the lower diagonal probabilities will certainly lead them to zero as we move on the time scale. We may not commit a serious mistake by treating the lower diagonal elements as zero when it reaches zero leaving the projection to subdiagonal and super diagonal elements. Construction of diagonal elements were done by using 11 and super diagonal by using 12 treating the subdiagonal elements as zero.

Surprisingly, a close examination of the observed transition matrices for the period 1901-81 reveals that treating subdiagonal elements as zero even in short term projections may be workable. With importance in movements of cities to higher classes as time passes, the construction of super diagonal elements is enough in a closed model. Diagonal elements may be obtained by subtracting the super diagonal values from one, under the assumption of subdiagonal elements being zero. The projected transition matrices presented in Table 3c is based on the linear assumption, and in Table 3*d* is based on growth curve assumption.

Table 4 provides the expected distribution of number of places by different Size classes under different assumptions of transition probabilities. Each transition probability matrix was used to project the CSD upto 2001. The observed distribution upto 1981 presented in Table 1 can be used for comparison purposes. The estimates under stationary assumption using individual transition matrices are presented in Tables 4a (i) to *4a* (vii). The subscript of *C* gives the

TABLE 4—EXPECTED DISTRIBUTION OF URBAN PLACES BY SIZE TAMILNADU : 1901-2001

4a. EXPECTED DISTRIBUTION OF PLACES BY SIZE : STATIONARY TRANSITION MATRICES OF OBSERVED MOVEMENTS

4a (') $C_{1901}P_{19}$oi-ii L, t=l,2, . . . , 10						to 00 C_{191}i Pjim-ai, t= 1,2,. . .9				
B	C	D	E	F		B	C	D	E	F
31	49	22	6	3	1911					
27	48	27	6	3	1921	31	48	20	9	3
24	46	31	1	3	1931	29	47	20	12	3
21	44	35	9	3	1941	28	46	19	15	3
19	41	39	9	3	1951	27	45	18	18	3
17	39	42	10	3	1961	26	44	17	21	3
16	36	45	11	3	1571	25	43	17	23	3
14	34	48	12	3	1981	24	42	16	26	3
12	32	51	13	3	1991	23	41	16	28	3
11	29	54	14	3	2001	13	34	36	15	13
4a (iii) C_{1821} $P^{t}_{1921-31}$, T=i, 2, ... 8						4a (iv) C_{l93l} $P^{t}_{1931-41}$. t=1.2. . . . 7				
22	47	29	9	4	1931					
16	43	37	10	5	1941	20	45	31	10	5
12	38	43	12	6	1951	18	43	33	11	6
9	33	48	14	7	1961	17	41	34	12	7
7	28	51	16	9	1971	16	39	35	13	8
5	23	54	16	13	1981	15	37	36	14	9
4	19	55	20	13	1991	14	35	36	15	11
3	16	56	21	15	2001	13	34	36	15	13

4a (v) C_{1941} $P^{t}_{1941-51}$, $t=1, 2, \ldots, 6$

10	38	45		
6	28	55		9
3	20	60		12
2	13	62		15
1	9	61		19
1	6	58		23

4a (vi) C_{1951} $P^{t}_{1951-61}$, $i=1, 2, \ldots$

1951					
1961	5	39	40	18	9
1971	3	37	36	23	12
1981	2	34	33	26	16
1991	1	31	30	28	21
2001	1	27	28	29	26

4a (vii) C_{1961} $P^{t}_{1961-71}$ $t = 1, 2, \ldots, 4$

1971	2	29	42	22	16
1981	1	20	40	25	25
1991	0	14	36	26	35
2001	0	9	31	26	45

4 B— EXPECTED DISTRIBUTION OF PLACES BY SIZE TRANSITION MATRICES OF AVERAGE MOVEMENTS

4b(i) $C_{1901} P^{t}_{1901-31}, t = 1, 2, \ldots, 10$						4b(iii) $C_{1901} P^{t}_{1901-71}, t = 1, 2, \ldots 10$				
B	C	D	E	F		B	C	D	E	F
30	49	22	7	3	1911	24	5	24	7	4
26	48	26	6	3	1921	17	50	30	9	5
23	46	30	8	4	1931	13	46	35	11	6
20	44	33	10	4	1941	10	41	39	14	7
18	41	36	12	4	1951	8	36	41	17	9
16	38	38	14	5	1961	7	31	42	19	12
14	35	40	16	6	1971	6	27	42	21	15
13	33	41	18	6	1981	4	21	41	24	21
11	31	42	20	7	1991	5	24	42	22	18
10	28	43	22	8	2001	4	18	40	25	24
4b(ii) $C_{1941} P^{t}_{1941-74}, t = 1, 2 \ldots 6$						4b(iv) $\bar{C}_{1941} P^{t}_{1902-71}, t = 1, 2, \ldots 13$				
9	44	38	13	7	1951	14	43	35	12	7
4	37	43	17	10	1961	10	39	38	15	9
2	29	45	21	14	1971	8	35	40	17	11
1	22	44	24	20	1981	6	31	41	19	14
1	16	42	26	26	1991	5	27	41	21	17
1	12	38	27	33	2001	4	23	41	23	20

TABLE 5—EXPECTED DISTRIBUTION OF URBAN PLACES BY SIZE : TAMILNADU : 1901-2001
NONSTATIONARY TRANSITION MATRICES OF OBSERVED MOVEMENTS

(a) Linear ProProjection 5(i) $C_{1911}P(t)$; t = 1.2, 9											(6) Growth Curves 5(ii) C_{1921} $P(t)$, t=1, 2, . . . ,,8									
B		C		D		E		F			B		C		D		E		F	
21		48		27		6		3		1921										
26		47		25		10		3		1931	30		47		19		12		3	
19		45		33		10		4		1941	22		46		28		11		4	
17		43		35		11		5		1951	20		44		30		12		5	
9		35		48		12		7		1961	10		37		44		13		7	
5		36		42		19		9		1971	5		38		39		20		9	
2		27		43		23		16		1981	2		28		41		23		17	
a	b	a	b	a	b	a	b	a	6		a	b	a	b	a	b	a	b	a	b
6	1	28	19	31	41	12	23	34	27	1991	6	1	28	20	30	40	11	22	36	28
2	0	25	13	36	36	9	23	39	39	2001	2	0	25	13	35	36	9	22	40	40
5(iii) C_{1931} $P(t)$; t = 1,2,. . .,7											5(iv) C_{1941} PC), i(=1, 2, . . .,6									
16		43		37		10		5		1941										
15		41		37		12		6		1951	18		43		33		11		6	
8		33		49		13		8		1961	9		36		46		12		8	
4		34		43		20		10		1971	5		37		40		19		10	
2		25		42		24		18		1981	2		28		41		23		17	
5	1	27	18	30	40	12	23	37	29	1991	6	1	28	20	30	40	11	22	36	28
1	0	32	12	35	35	9	23	42	41	2001	2	0	25	13	35	36	9	22	40	40

Table 5 *(contd. on page* 34)

Table 5 *(contd.from* ***page*** **33)**

5(v) $C_{1951}P(t)$, $t = 1, 2, \ldots 5$

	B		C		D		E		F	
1961	6		28		55		13		9	
1971	3		28		47		22		11	
1981	1		21		43		26		20	
1991	4	0	24	15	30	39	12	25	41	32
2001	1	0	21	9	34	33	9	24	46	45

5(vi) $C_{1961}P(t)$, $t=1,2,\ldots 4$

	B		C		D		E		F	
1971	3		37		36		23		12	
1981	1		27		38		24		21	
1991	6	0	26	19	28	37	11	22	40	33
2001	2	0	24	11	33	33	8	22	44	45

5(vii) $C_{1971}P(t)$, $t= 1,2, \ldots,3$

	B		C		D		E		F	
1981	1		20		40		25		25	
1991	4	0	23	14	28	37	11	23	45	37
2001	2	0	20	9	32	31	8	22	49	49

year of initial distribution used in building the estimates. The superscript of P indicates the number of projection while the suffix of P^t indicates the period of transition matrix used to obtain the estimates under stationary assumption. For example $C_{1921}\ P_{1921-31}$ represent the estimates for 1981 using the transition matrix constructed for 1921-31 with initial distribution of 1921 under stationary assumption.

The difference between the observed and the expected distribution and the difference between two expected distributions can be had for any combination of projections. Any projected figure can be compared with the figures given in Table 1 to yield the difference between the observed and the expected. Any projected figure can also be compared to any other projected figure for the same time period under different transition matrices. For example $C_{im}iP'_{gn}-_{a}i$, $C_{1951}P^3_{1951-61}$, $C_{1901}P^8_{1901-31}$, and $C_{1941}P^4_{1941-71}$ provides estimates for 1981 under different transition matrices providing comparison of estimated figures for the same point of time under varied transition matrices.

The estimates based on $P_{1901-11}$ transition probabilities with 1901 as initial distribution $C_{1901}\ P^t_{1901-11}$, $(t = 1, 2, \ldots, 10)$ provides a good approximation to the observed CSD upto 1941 and thereafter the differences are striking. 1911-21 transition probabilities provide poor estimates even in the earlier decades. With reference to 1921-31 transition matrix estimates upto 1961 are highly satisfactory but with greater differences compared to 1971. Surprisingly a positive bias invariably occurs in the lower size classes. This indicates the insensitiveness of the transition probability matrix to the reality of the situation that greater movements are taking place in the lower strata and transitions are gaining momentum after 1921.

1921 is considered to be the dividing point in the growth of Tamil Nadu population. A slowly growing population upto 1921 started growing rapidly after 1921. To bring to the realities of the situation under a stationary assumption, transition matrices constructed on the average movements (Table *3b)* were used to obtain the estimates of CSD and are presented in Tables *4b* (i) to *4b* (iv). It is easy to see that the differences in the upper end of the distribution between the observed and the expected city size distribution is considerably reduced.

The estimates constructed under nonstationary assumption are given in Table 5. While generating estimates beyond the observed point under nonstationary operation of the transition matrices two sets of estimates are given for 1991 and 2001 under *'a'* and *'b'* The figures under 'a' give the estimates conss-tructed under nonstationary assumption using linearly projected transition matrices and the figures under *'b'* provide the estimates under the operation of transition matrices under growth curve assumption.

There are 69 cities with size 20,000 and below in 1931 and 18 cities in 1981. Under the stationary assumption of $P_{1901-11}$ with 1901 as base, we get 70 and 48 cities with size below 20,000 in 1931 and 1981. This series also yields 40 cities **of** same size in 2001. Under the nonstationary assumption, we get 73 and 29 cities in 1931 and 1981 under similar set up. It is evident that the nonstationary condition provides a good approximation to the anticipated long run projections. This series also yields 27 and 13 cities at 2001 under linear and growth curve assumptions. 1981 census had 93 cities with size above 20,000. $P_{1941-51}$ with C_{1941} under the stationary assumption give 96 cities and the nonstationary assumption gives 81 cities, greatly improving the estimates. The series under nonstationary model provides 84 and 98 cities at 2001 under linear and growth curve assumptions respectively. Information contained in Tables 4 and 5 can be used to develop useful comparisons of estimates under any required set up.

The chi-square test for goodness of fit were carried out between observed and expected CSD through 1901-81. Based on this, the CSD at 2001 under linear and growth curve assumptions is fixed corresponding to the least chi-square value. Treating this CSD at 2001 as having been realised, the chi-square tests were carried out with realization at 2001 with various transition matrices. The results of this exercise lead to the inference that projections based on growth curve assumptions are more likely to be realised.

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